The Poisson Distribution

Gary Schurman MBE, CFA

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The Poisson distribution is a discrete probability distribution where a Poisson-distributed random variable can take the integer values $[0, 1, 2, 3, ..., \infty]$. The distribution is bounded on the left by zero and on the right by infinity. The Poisson distribution gives us the probability of realizing a given number of events over a fixed interval where the interval is defined as time, area, distance, etc. With the Poisson distribution there is only one parameter to estimate - the average rate of event occurrence.

In mathematical terms the Poisson distribution is the limit of the binomial distribution as population size n goes to infinity. With the binomial distribution we want to determine the probability of realizing k events out of a population of n possible events (i.e. population size is bounded). With the Poisson distribution we want to determine the probability of realizing k events out of a population of infinite size (i.e. population size is unbounded). In the sections that follow we will develop the mathematics for the Poisson distribution and then use the mathematics to solve a hypothetical problem.

Our Hypothetical Problem

ABC Company insures properties along the Gulf Coast of the United States. Statutory reserves are liabilities that the insurance company is legally required to maintain on its balance sheet with respect to expected future claims against the company for insured losses. To assist ABC Company in setting its reserves we are asked to provide actuarial guidance. Given that the historical average rate of occurrence of a major hurricane (i.e. Category Three or higher) along the Gulf Coast is two per year what are the probabilities of zero to ten major hurricanes over a one year time period?

The Poisson Distribution as the Limit of the Binomial Distribution

Using the binomial distribution the equation for the binomial probability (B) of realizing k events out of a finite population of n possible events is...

$$B \binom{n}{k} = \frac{n!}{k!\,(n-k)!}\, p^k\,(1-p)^{n-k} \tag{1}$$

What is the probability of realizing k events out of a population of infinite size? The Poisson distribution is the limit of the binomial distribution (as defined by Equation (1) above) as population size n goes to infinity. The equation for the Poisson probability (P) of realizing k events out of a population of infinite size is therefore...

$$P(k) = \lim_{n \to \infty} B \binom{n}{k} = \lim_{n \to \infty} \frac{n!}{k!\,(n-k)!}\, p^k\,(1-p)^{n-k} \tag{2}$$

Per the binomial distribution the number of events that we expect to be realized (i.e. the mean) is...

$$mean = \mathbb{E}\left[k\right] = n\,p \tag{3}$$

As population size n goes to infinity the expectation as defined by Equation (3) no longer is valid because infinity times p is equal to infinity. We therefore need to redefine the probability p. We will define λ to be the number of events that we expect to be realized. The equation for λ is...

$$\lambda = n\,p \text{ ...such that... } p = \frac{\lambda}{n} \tag{4}$$

After substituting Equation (4) for p in Equation (2) above the probability of realizing k events out of a population of infinite size is...

$$P(k) = \lim_{n \to \infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n \to \infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$= \lim_{n \to \infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-k} \tag{5}$$

Using Appendix Equation (23) we can rewrite Equation (5) above as...

$$P(k) = \lim_{n \to \infty} \left(\prod_{i=0}^{k-1} n - i\right) n^{-k} \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-k} \tag{6}$$

Using Appendix Equations (24), (26) and (27) we can rewrite Equation (6) above as...

$$P(k) = n^k \, n^{-k} \frac{\lambda^k}{k!} \, e^{-\lambda} \, (1) \tag{7}$$

If we multiply the terms on the right-hand side of Equation (7) the equation for the probability of realizing k events out of a population of infinite size using the Poisson Distribution is...

$$P(k) = \frac{\lambda^k \, e^{-\lambda}}{k!} \tag{8}$$

The Mean and Variance of the Poisson Distribution

The Poisson distribution is a distribution of discrete random variables. If the random variable k is discrete then the moment generating function $(M(t))$ of the random variable k where $P(k)$ is the probability mass function (as defined by Equation (8) above) can be written as...

$$M_k(t) = \sum_{k=0}^{\infty} e^{tk} P(k) \tag{9}$$

If we combine Equations (8) and (9) we can write the equation for the moment generating function of the Poisson distribution as...

$$M_k(t) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k \, e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t)^k \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \tag{10}$$

Noting that the definition of the exponential function is...

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{11}$$

If we define z in Equation (11) to be...

$$z = \lambda e^t \tag{12}$$

Then Equation (10) becomes...

$$M_k(t) = e^{-\lambda} e^z$$

$$= e^{z-\lambda}$$

$$= e^{\lambda e^t - \lambda}$$

$$= e^{\lambda(e^t - 1)} \tag{13}$$

Equation (13) above is the moment generating function for the Poisson distribution and we can use it to calculate the distribution's mean and variance.

We can derive the first moment of the Poisson distribution by setting $t = 0$ in Appendix Equation (30). The first moment of the Poisson distribution is...

$$\mathbb{E}\left[k\right] = \lambda\,e^{\lambda(e^0-1)+0} = \lambda \tag{14}$$

We can derive the second moment of the Poisson distribution by setting $t = 0$ in Appendix Equation (33). The second moment of the Poisson distribution is...

$$\mathbb{E}\left[k^2\right] = \lambda\left(\lambda\,e^0 + 1\right)e^{\lambda(e^0-1)+0} = \lambda^2 + \lambda \tag{15}$$

Using Equations (14) and (15) above the mean and variance of the Poisson distribution are...

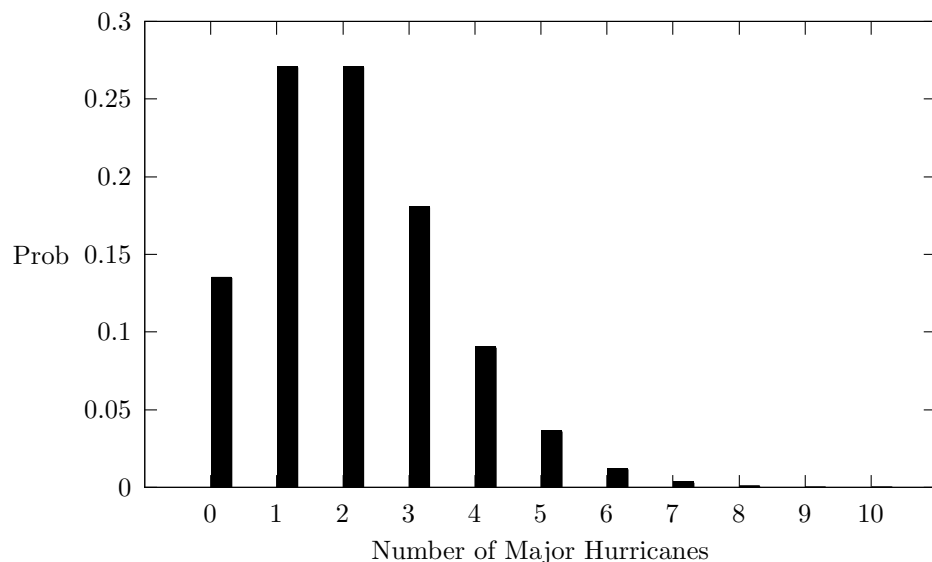$$mean = \mathbb{E}\left[k\right] = \lambda \tag{16}$$

$$variance = \mathbb{E}\left[k^2\right] - \left[\mathbb{E}\left[k\right]\right]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \tag{17}$$

The Answer to the Hypothetical Problem

As noted above, the only parameter that we need to estimate in order to use the Poisson distribution is the average rate of event occurrence which is...

$$\lambda = \text{Average Rate of Occurrence} = 2.00 \tag{18}$$

The graph and table below present the probabilities of a major hurricane over a one year time interval...

Number	Probability
0	0.13534
1	0.27067
2	0.27067
3	0.18045
4	0.09022
5	0.03609
6	0.01203
7	0.00344
8	0.00086
9	0.00019
10	0.00004
Total	0.99999

For example, the probability for three major hurricanes using Equation (8) above is...

$$P(k) = \frac{2^3 \times exp(-2)}{3!} = 0.18045 \tag{19}$$

Appendix

A. The equation for n factorial is...

$$n! = \prod_{i=0}^{n-1} n - i \tag{20}$$

B. Given that k is an integer value less than n we can substitute $n - k$ for n in Equation (20) such that the equation for $n - k$ factorial is...

$$(n - k)! = \prod_{j=0}^{n-k-1} n - k - j \tag{21}$$

C. Using Equations (20) and (21) above we can rewrite the equation for n factorial as...

$$n! = \prod_{i=0}^{k-1} n - i \prod_{j=0}^{n-k-1} n - k - j \tag{22}$$

D. Using Equations (21) and (22) above the equation for n factorial divided by $n - k$ factorial is...

$$\frac{n!}{(n - k)!} = \prod_{i=0}^{k-1} n - i \tag{23}$$

E. The limit of Equation (23) as n goes to infinity is...

$$\lim_{n \to \infty} \prod_{i=0}^{k-1} n - i = \prod_{i=0}^{k-1} n = n^k \tag{24}$$

Note that as n goes to infinity the value of i becomes less significant such that the limit of $n - i = n$.

F. As we divide the period over which we earn a return into an infinite number of compounding periods it can be shown that...

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r \tag{25}$$

G. Using Equation (25) above the limit of the quantity one minus lambda divided by n to the nth power as n goes to infinity is...

$$\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = \lim_{n \to \infty} \left(1 + \frac{-\lambda}{n}\right)^n = e^{-\lambda} \tag{26}$$

H. The limit of the quantity one minus lambda divided by n to the $-k$th power as n goes to infinity is...

$$\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{-k} = 1^{-k} = 1 \tag{27}$$

I. The first derivative of the moment generating function as defined by Equation (13) above with respect to t is...

We will define θ as...

$$\theta = \lambda\left(e^t - 1\right) \text{ ...such that... } \frac{\delta\theta}{\delta t} = \lambda\,e^t \tag{28}$$

Using this definitions we can rewrite Equation (13) as

$$M_k(t) = e^\theta \tag{29}$$

The first derivative of the moment generating function as redefined by Equation (29) is...

$$\begin{aligned}
M'_k(t) &= \frac{\delta M_k(t)}{\delta t} \\
&= \frac{\delta M_k(t)}{\delta\theta}\frac{\delta\theta}{\delta t} \\
&= e^\theta \times \lambda\,e^t \\
&= \lambda\,e^{\theta+t} \\
&= \lambda\,e^{\lambda(e^t-1)+t}
\end{aligned} \tag{30}$$

J. The second derivative of the moment generating function as defined by Equation (13) above with respect to t is...

We will define θ as...

$$\theta = \lambda(e^t - 1) + t \text{ ...such that... } \frac{\delta\theta}{\delta t} = \lambda\,e^t + 1 \tag{31}$$

Using this definitions we can rewrite Equation (30) as

$$M'_k(t) = \lambda\,e^\theta \tag{32}$$

The derivative of Equation (32) with respect to t is...

$$\begin{aligned}
M''_k(t) &= \frac{\delta M'_k(t)}{\delta t} \\
&= \frac{\delta M'_k(t)}{\delta\theta}\frac{\delta\theta}{\delta t} \\
&= \lambda\,e^\theta \times \left(\lambda\,e^t + 1\right) \\
&= \lambda\left(\lambda\,e^t + 1\right)e^{\lambda(e^t-1)+t}
\end{aligned} \tag{33}$$